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TECHNICAL NOTE 3343

SUBSONIC EDGES IN THIN-WING AND SLENDER-BODY THEORY

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SUMMARY

A simple technique is presented for correcting the results of thin-wing theory near round or sharp edges at which the normal component of free-stream velocity is subsonic. The flow in planes normal to such edges is actually brought to rest, but thin-wing theory instead predicts infinite velocities. Furthermore, corresponding to the circulatory flow around leading edges, thin-wing theory predicts additional singularities that do not actually exist for round edges and exceed their true strength for sharp edges. All these singularities grow worse if one formally attempts to improve the solution by iteration. To correct these defects, the formal thin-wing solution is here rendered uniformly valid by comparing with exact solutions for simple profiles having the same nose shape. In this way Lighthill's rule for correcting the surface speed on round-nosed airfoils in incompressible flow is extended to higher approximations, to compressible flow, to three-dimensional wings, and to sharp edges. Corresponding results for slender bodies of revolution are considered briefly; in particular, the rules for round-nosed airfoils in incompressible flow are shown to apply also to round-nosed bodies of revolution.

INTRODUCTION

The linearized theory of thin wings, which has proved so fruitful for both subsonic and supersonic flows, is known to fail near leading and trailing edges if the component of free-stream velocity normal to the edge is subsonic. The flow in a plane normal to such an edge is actually brought to rest at some point near the edge, but linearized theory predicts infinite velocities instead.[1] Furthermore, there is flow around the leading edge except at one angle of attack, and linearized theory then predicts an additional singularity that does not actually exist for round edges and exceeds its true strength for sharp edges.

These local failures are sometimes of little practical consequence, since the singularities are integrable and the lift is determined correct to a first approximation. However, it has long been known that special

[1] At round edges, linearized theory predicts only finite speeds on the surface, but the velocity of the stream approaching the stagnation point becomes infinite.

attention must be paid to the singularity in calculating the drag of thin lifting wings (ref. 1). More recently, R. T. Jones has pointed out that the same is true for nonlifting wings and has shown how to account for the singularity in evaluating drag (ref. 2). In any case, the calculated velocities and pressures are incorrect near such an edge.

If one attempts to refine the theory by formally iterating upon the first approximation, the local failure leads to serious difficulties. In incompressible flow, the singularities are found to increase in strength at each stage of the iteration, so that the solution, though improved elsewhere, becomes increasingly inaccurate near the edge. Moreover, for round edges these higher terms give nonintegrable contributions to the aerodynamic forces. Even more serious difficulties may arise in compressible flow, where the infection spreads in some cases, so that the formal second approximation is incorrect not only at the edges but over the entire airfoil surface.

The remedy for all these difficulties is clearly to base the iteration upon a first approximation that is valid even near the edges. This would appear to require abandoning at the outset the assumption of small disturbances, which so greatly simplifies analysis. Fortunately, however, Lighthill has found (ref. 3) that for round edges in incompressible flow all the results of small-disturbance theory can be salvaged. He shows that a very simple rule converts the formal second-order solution of thin-airfoil theory into an approximation that is uniformly valid near the edge.

Lighthill's result follows from application of his general technique for rendering approximate solutions to physical problems uniformly valid (ref. 4). This consists in determining progressively with each approximation a straining of the coordinates that ensures uniform convergence. For a round-nosed airfoil in incompressible flow the straining is, to the first approximation, a contraction of abscissas that shifts the leading edge by half its radius, as was noted by Munk in 1922 (ref. 5). The uniformly valid solution is then the same function of the strained coordinates as the thin-airfoil solution was of the unstrained coordinates.

It is not yet clear how far Lighthill's technique can be extended. Fox has indicated (ref. 6) that it fails for higher approximations, other nose shapes, or compressible flows, but in a recent series of seminars at California Institute of Technology, Tsien suggested that proper application of the technique will lead to success. In any case, a quite different technique is used here, which is particularly suited to the study of edges, and for that special problem is simpler than Lighthill's general procedure. It consists in comparing the exact and thin-wing solutions for simple shapes that approximate the airfoil in the vicinity of the edge. Thus Lighthill's rule for round noses is reproduced by considering incompressible flow past a semi-infinite parabola. This comparison technique is then readily extended to higher approximations, compressible flows, three-dimensional wings, sharp edges, and slender bodies of revolution. Attention is confined here to the practical matter of calculating

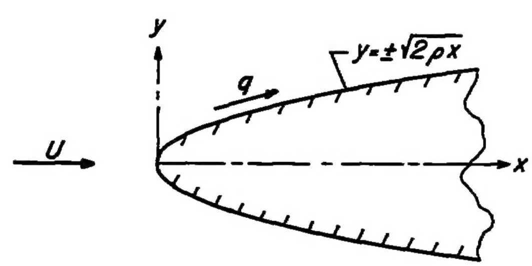
surface speeds, but the whole flow field can be corrected by the same method. The technique is applied here only to edges that are round (finite radius of curvature) or sharp (finite slope), since these occur on most existing profiles. However, the procedure can, in principle, be extended to other edge shapes and to slender bodies of other than circular cross section.

The analysis of this paper is intuitive in spirit. Where convenient, some of the results have been deduced by appeal to physical reasoning. It is presumed that at these points the analysis could be bolstered by more rigorous mathematical arguments.

The author is indebted to R. T. Jones for many helpful discussions throughout the course of this work.

ROUND-NOSED AIRFOILS IN INCOMPRESSIBLE FLOW

The technique to be used can be introduced by restricting attention first to an uncambered two-dimensional airfoil at zero angle of attack in incompressible flow. Thin-airfoil theory breaks down in the vicinity of the leading edge. There, a round-nosed airfoil can be approximated by a parabola. Accordingly, consider the exact velocity distribution over a parabola of leading-edge radius ρ in a uniform stream parallel to its axis (sketch (a)). The flow speed on the surface can be found by conformal mapping, or simply by Munk's rule (ref. 7) that the surface velocity on any ellipsoid subjected to uniform flow along a principal axis is the projection of the maximum velocity onto the plane tangent to the surface. For a parabola the maximum is the free-stream velocity, so that with x measured downstream from the nose the surface speed ratio is



Sketch (a)

$$\frac{q}{U} = \left(\frac{x}{x + \rho/2}\right)^{1/2} \tag{1}$$

(All symbols are defined in the Appendix.) Expanding this expression for small ρ/x yields the formal series

$$\frac{"q"}{U} = 1 - \frac{\rho}{4x} + \ldots \tag{2}$$

and this must be the series that would be given by thin-airfoil theory. (It is actually the second-order thin-airfoil solution - the first-order term being zero for a parabola - since the nose radius of any profile varies with the square of its thickness.) It is clear how the spurious singularities of thin-airfoil theory arise at the leading edge.

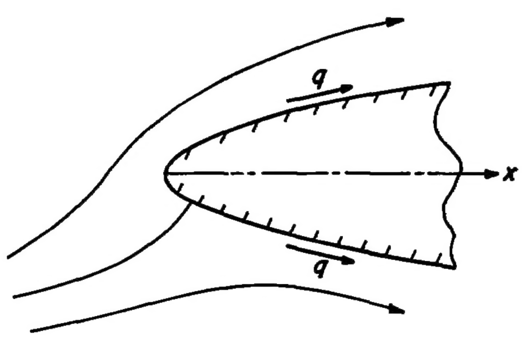
The formal second-order thin-airfoil solution "q_2" for a parabola can be converted into an approximation $\bar{q}$ that is uniformly valid by multiplying it by the ratio of the exact result to the series expansion:

$$\frac{\bar{q}}{U} = \left[\frac{\left(\dfrac{x}{x + \rho/2}\right)^{1/2}}{1 - \dfrac{\rho}{4x} + \ldots}\right]\frac{"q_2"}{U} \tag{3a}$$

Simplifying this insofar as possible, retaining only terms of order ρ/x, gives the rule

$$\frac{\bar{q}}{U} = \left(\frac{x}{x + \rho/2}\right)^{1/2}\left(\frac{"q_2"}{U} + \frac{\rho}{4x}\right) \tag{3b}$$

It can now be argued that this rule, derived for a parabola, applies with some degree of approximation to any uncambered airfoil of finite leading-edge radius ρ at zero angle of attack. Near the leading edge, where the flow disturbances are large, the exact speed on any airfoil is nearly that on a parabola of the same nose radius; hence, the rule provides a first approximation to the disturbances there. Far from the edge, the disturbances are small and, in general, different from those for a parabola; there the ratio in equation (3a) is essentially unity, so that the rule reproduces the second-order solution for the airfoil, as it should. Hence, the rule yields an approximation that is uniformly valid. (It is not, of course, valid near a second edge; combined edges will be considered later.)

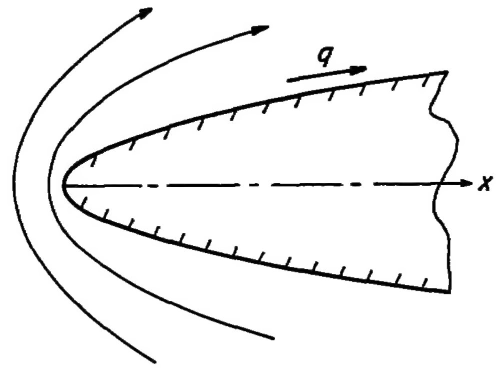


Sketch (b)

The rule clearly applies also to cambered airfoils at the "ideal" angle of attack for which the stagnation point coincides with the vertex. However, at other angles of attack the flow field includes a component of circulatory flow around the edge (sketch (b)). Because of the linear nature of incompressible flow, this component can be treated separately. The amount of this component is proportional to the sine of the angle of

attack α_o measured from the ideal angle. The thin-airfoil series for this component can be corrected by comparison with the circulatory flow past a parabola (sketch (c)). Conformal mapping or Munk's rule[2] gives the speed on the upper and lower surfaces of the parabola (counted positive away from the vertex) as

$$\frac{q}{U} = \pm k \sin \alpha_o \left(\frac{1}{x + \rho/2}\right)^{1/2} \tag{4}$$



Sketch (c)

Here the factor k is a coefficient of order unity that relates the circulatory flows on the parabola and the actual airfoil; it depends on the entire airfoil shape (and the trailing-edge condition), but its value is not required here. Expanding this expression for small ρ/x yields the formal thin-airfoil series for the circulatory component, which is, to second order in thickness and angle of attack,

$$\frac{"q_2"}{U} = \pm\frac{k\alpha_o}{\sqrt{x}} \tag{5}$$

As before, the ratio of the exact result to its series expansion is a factor that serves to correct the thin-airfoil solution for any round-nosed airfoil. Hence the rule for the circulatory component is

$$\frac{\bar{q}}{U} = \left(\frac{x}{x + \rho/2}\right)^{1/2} \cdot \frac{"q_2"}{U} \tag{6}$$

Now it happens that this is essentially the same as the rule for the other component (eq. (3b)). Consequently, the two rules can be combined. It follows that the rule of equation (3b) applies to a round-nosed airfoil at any angle of attack. It is, in fact, precisely Lighthill's rule, which he found using his technique for rendering approximate solutions uniformly valid.

The accuracy of this rule near the nose of an airfoil depends upon how closely the nose is approximated by a parabola. Lighthill has

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[2]Munk's rule can be used here by recognizing that the circulatory flow about a parabola is identical with the limit of the crossflow past an ellipse as its thickness ratio vanishes.

considered airfoils described by analytic arcs, which means that near the leading edge

$$y = \epsilon(\pm F_0 x^{1/2} + C_1 x \pm F_1 x^{3/2} + \ldots) \tag{7}$$

ϵ being a small parameter and the F_n and C_n coefficients of order unity associated with thickness and camber, respectively; also, the angle of attack is assumed to be of order ϵ. Now the example of the parabola shows that the problems of nonuniformity are confined to a region proportional to the leading-edge radius, that is, in which x is $O(\epsilon^2)$. In this region the analytic airfoils of equation (7) coincide with parabolas except for terms of $O(x^{1/2})$, which is $O(\epsilon)$. It seems reasonable to assert that the surface speed will be approximated to the same order of accuracy as is the airfoil shape. This means that applying the rule to the second-order thin-airfoil solution yields a uniformly valid solution that is in error by $O(\epsilon)$ near the leading edge (though of course only by $O(\epsilon^3)$ far away). This agrees with Lighthill's conclusion, based on more rigorous analysis. However, initially uncambered airfoils (for which $C_1 = 0$) are parabolic except for terms of $O(x) = O(\epsilon^2)$, so that in this case the solution rendered uniformly valid is in error by at most $O(\epsilon^2)$ near the nose.

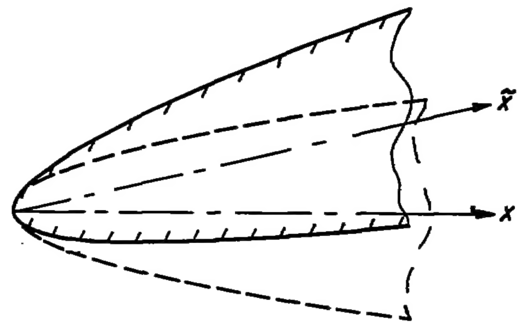
It would at first sight seem appropriate to say that Lighthill's rule yields a solution for analytic airfoils correct only to zero order near the nose (first order in the absence of camber), since only terms of $O(1)$ are found correctly, and this is the order of the undisturbed stream. However, this occurs where the perturbations themselves are $O(1)$ - for example, the surface speed one leading-edge radius from the nose is 0.707 of the free stream - so that the leading term in the velocity disturbance (and hence in the pressure coefficient) is given correctly. With this in mind, we shall say that Lighthill's rule renders the thin-airfoil solution valid to first order near the nose for cambered airfoils and to second order for initially uncambered ones.

If only first-order terms are retained, Lighthill's rule becomes simply

$$\frac{\bar{q}_1}{U} = \left(\frac{x}{x + \rho/2}\right)^{1/2} \left(\frac{"q_1"}{U}\right) \tag{8}$$

This converts the linearized solution into a uniformly valid first approximation. Results differing from this only by higher-order terms were given earlier by Goldstein (Approximation II of ref. 8) and by Riegels (ref. 9). It will be seen later that Riegels' form happens to be both simpler and more accurate than Lighthill's.

It seems desirable to refine the rule so that it is correct to second order - that is, to $O(\epsilon^2)$ - even near the nose of a cambered airfoil. This is readily done by considering flow past a "cambered" or sheared parabola (sketch (d)) given by the first two terms of equation (7). Shearing transforms a parabola into another parabola whose vertex and nose radius are changed only to higher order. Hence the previous rule of equation (3b) applies if x is replaced by the oblique abscissa $\tilde{x}$. Now it is easily shown that for a point on the surface of the parabola



Sketch (d)

$$\tilde{x} = x \pm \epsilon C_1 \sqrt{2\rho x} + \ldots \tag{9}$$

where ϵC_1 is the initial slope of the camber line. Hence the rule becomes

$$\frac{\bar{q}_2}{U} = \left(\frac{x \pm \epsilon C_1 \sqrt{2\rho x}}{x + \rho/2 \pm \epsilon C_1 \sqrt{2\rho x}}\right)^{1/2} \left(\frac{"q_2"}{U} + \frac{\rho}{4x}\right) \tag{10}$$

where, again, the upper and lower signs pertain to the upper and lower surfaces of the airfoil.

This rule, which renders the thin-airfoil solution uniformly valid to second order, is probably all that will ever be required in practice. However, it can easily be extended still further if necessary. For example, consider an uncambered analytic airfoil. Its nose can be approximated by an ellipse except for terms of $O(x^{5/2})$, which is $O(\epsilon^5)$ in the region of interest. Hence, the following rule, derived from consideration of the exact flow past an ellipse, renders the thin-airfoil solution uniformly valid to fourth order:[3]

$$\frac{\bar{q}_4}{U} = \frac{1}{\sqrt{1 + \lambda}} \left(\frac{"q_4"}{U} + \frac{\lambda}{2} \frac{"q_2"}{U} - \frac{1}{8}\lambda^2\right), \quad \lambda = \frac{\epsilon^2}{4} \left(\frac{{F_o}^2}{x} + 6F_oF_1\right) \tag{11}$$

The fourth-order result for cambered airfoils can be obtained from this using the fact that a sheared ellipse is an ellipse; this will be given later when alternative forms are considered.

[3]An ellipse corresponds to $F_1 < 0$; application of this rule to cases for which $F_1 > 0$ rests on an appeal to the principle of analytic continuation.

From these rules, corresponding rules follow at once for any of the velocity components. For example, Lighthill's rule for the horizontal velocity component at the surface can be deduced directly from equation 3(b):

$$\frac{\bar{u}}{U} = \frac{x}{x + \rho/2}\left(\frac{"u_2"}{U} + \frac{\rho}{2x}\right) \tag{12}$$

In calculating pressures from velocities, the full Bernoulli equation must be used since the disturbances are not small.

REMARKS ON THE METHOD

The essence of the technique used here is to consider the exact solution for some simple shape that approximates the airfoil in the vicinity of its edge, where thin-airfoil theory breaks down. The formal thin-airfoil solution can then be corrected by multiplying it by the ratio of this locally exact solution to its series expansion, simplifying insofar as possible. This means that the rules are essentially multiplicative in nature. The process of simplification tends to conceal this fact, so that they appear to be at least partially additive.

The multiplicative nature of the rules is apparent from the viewpoint of conformal mapping. Thin-airfoil theory gives a formal series for the mapping of the profile onto a straight line, which fails near a round nose. With Lighthill's rule, thin-airfoil theory is relied upon only to map the profile onto an initially osculating parabola. The radical in equation (3b) is the speed on the parabola; the remaining factors represent the speed ratio associated with this intermediate mapping, by which it is to be multiplied.

From another point of view, Lighthill's rule implies that the airfoil is represented by sources distributed over the surface of a parabola, rather than along a straight line. To second order, the perturbations induced by the sources are proportional to the local speed of the basic flow past a parabola, which indicates again the multiplicative nature of the rules. This viewpoint remains valid in the subsequent extensions to subsonic and axially symmetric flows, where conformal mapping is inapplicable.

ROUND-NOSED AIRFOILS IN SUBSONIC FLOW

The preceding results can be extended to compressible flow by using solutions for subsonic flow past parabolas and ellipses. Although no exact solutions are known, it seems that adequate approximations can be found so long as the flow is purely subsonic.

At stagnation points in subsonic flow, thin-airfoil theory predicts infinities of the same order as in incompressible flow. Although these infinities can be avoided by using the exact condition of tangent flow at the surface (which in incompressible flow gives the true solution), the result is nevertheless incorrect near stagnation points. Thus, Hantzsche and Wendt (ref. 10) work with the perturbation stream function and impose the exact tangency condition by means of conformal mapping, but even their first approximation gives not zero speed at stagnation points but a velocity $u/U = -M^2(1 - M^2)^{-1/2}$. This is primarily a defect associated with the stream function, but even using the velocity potential in the same way gives results which are no more than qualitatively correct near stagnation edges.

An alternative procedure for treating subsonic flows by successive approximations is the Janzen-Rayleigh expansion in powers of M^2. The two methods are known to complement each other, being, in fact, simply different series representations of the true solution. In particular, the Janzen-Rayleigh result is most accurate near stagnation points, where thin-airfoil theory breaks down. It is therefore suited for the present purpose.

Consider again an uncambered round-nosed airfoil at zero angle of attack. Probably the second-order thin-airfoil solution is the most that will be treated in practice; then it is enough to consider subsonic flow along the axis of a parabola (sketch (a)). Imai has recently calculated (ref. 11) the Janzen-Rayleigh solution for a parabola including terms in M^4, and has given the surface speed in essentially the form

$$\frac{q}{U} = f_o + M^2 f_1 + (\gamma + 1)M^4 f_2 + M^4 f_3 + O(M^6) \tag{13}$$

Here the f_n are increasingly complicated functions of x/ρ (where ρ is the nose radius), f_o being the incompressible result of equation (1).

Far behind the nose, Imai's result reduces to

$$\frac{q}{U} = 1 - \frac{\rho}{4x}\left(1 + M^2 + \frac{\gamma + 1}{4} M^4 + M^4\right) + O(M^6,\ \rho^2/x^2) \tag{14a}$$

This should agree to this order with thin-airfoil theory and indeed it does; second-order thin-airfoil theory (ref. 12) gives

$$\frac{"q_2"}{U} = 1 - \frac{\rho}{4x}\left[1 + M^2\frac{(\gamma + 1)M^2 + 4\beta^2}{4\beta^4}\right] + O(\rho^2/x^2) \tag{14b}$$

and the reduced result of Imai, equation (14a), is just the curtailed expansion of this in powers of M^2. Comparing these two expressions suggests a modification of the Janzen-Rayleigh solution that will include the full second-order thin-airfoil result:

$$\frac{q}{U} = Q\left(\frac{x}{\rho}, M\right) = f_0 + \frac{M^2}{\beta^2} f_1 + (\gamma + 1)\frac{M^4}{\beta^4} f_2 + M^4(f_3 - f_1) + O(M^6) \qquad (15a)$$

where, according to Imai's solution,[4]

$$\left.\begin{aligned}
f_0 &= \left(\frac{x}{x + \rho/2}\right)^{1/2} = \sin\frac{\theta}{2}, \qquad \text{where } \theta = \cos^{-1}\left(\frac{\rho - 2x}{\rho + 2x}\right) \\
f_1 &= -\frac{1}{4}\left(\cos\frac{\theta}{2}\right)^3\left[2\tan\frac{\theta}{2} - 2\sin\theta\log\left(2\cos\frac{\theta}{2}\right) + \theta\cos\theta\right] \\
f_2 &= -\frac{1}{32}\left(\cos\frac{\theta}{2}\right)^3\left[4\tan\frac{\theta}{2} - 13\sin\theta - 4(\sin\theta - 3\sin 2\theta)\right. \\
&\qquad \left.\log\left(2\cos\frac{\theta}{2}\right) + 2\theta(\cos\theta - 3\cos 2\theta)\right] \\
f_3 - f_1 &= -\frac{1}{32}\left(\cos\frac{\theta}{2}\right)^3\left[-\left(\frac{17}{2} - \frac{\pi^2}{3}\right)\sin\theta + \frac{\pi^2}{6}\sin 2\theta + 7\sin 2\theta\right. \\
&\qquad \log\left(2\cos\frac{\theta}{2}\right) - \frac{\theta}{2}(5 - 12\cos\theta + 7\cos 2\theta) + \\
&\qquad \frac{\theta^2}{8}(3\sin\theta + 8\sin 2\theta + 3\sin 3\theta) + \\
&\qquad \frac{1}{2}(\sin\theta - 4\sin 2\theta - 3\sin 3\theta)\log^2\left(2\cos\frac{\theta}{2}\right) + \\
&\qquad \frac{\theta}{2}(5\cos\theta + 8\cos 2\theta + 3\cos 3\theta)\log\left(2\cos\frac{\theta}{2}\right) - \\
&\qquad \left. 2(2\cos\theta + \cos 2\theta)\int_0^\theta \log\left(2\cos\frac{\varphi}{2}\right)d\varphi\right]
\end{aligned}\right\} \qquad (15b)$$

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[4]Here an error in sign in reference 11 has been corrected in the expression for f_1; the correction has been confirmed by Prof. Imai.

This modification seems to be a natural one; for example, the combination $(f_3 - f_1)$ is simpler than its original counterpart f_3. A short table of the function $Q(x/\rho,M)$ calculated from this approximation[5] is given below for $\gamma = 7/5$.

M \ x/ρ	0	0.25	0.5	1	2	5
0	0	0.5774	0.7071	0.8165	0.8944	0.9535
.4	0	.5554	.6871	.8007	.8831	.9474
.5	0	.5409	.6740	.7902	.8751	.9430
.6	0	.5211	.6556	.7746	.8624	.9355
.7	0	.4970	.6296	.7492	.8396	.9210
.8	0	.4888	.5993	.7023	.7887	.8844

Applying the argument used before to equations (14) and (15) shows that the second-order thin-airfoil solution for any uncambered analytic profile of nose radius ρ in subsonic flow at zero angle of attack is converted into a uniformly valid second approximation by the rule

$$\frac{\bar{q}_2}{U} = Q\left(\frac{x}{\rho}, M\right)\left[\frac{"q_2"}{U} + \frac{(\gamma + 1)M^4 + 4\beta^2}{4\beta^4}\frac{\rho}{4x}\right] \tag{16a}$$

Here $Q(x/\rho,M)$ is the exact subsonic solution for a parabola. It can, in principle be found to any desired accuracy since the Janzen-Rayleigh method is believed to converge for purely subsonic flows[6] and, for axial flow past a parabola, the flow is purely subsonic for any free-stream Mach number less than unity. For practical purposes, the M^4 approximation of equation (15) that is tabulated above will probably suffice. For application to linearized theory, this rule reduces simply to

$$\frac{\bar{q}_1}{U} = Q\left(\frac{x}{\rho}, M\right)\left(\frac{"q_1"}{U}\right) \tag{16b}$$

which yields a uniformly valid first approximation.

This last rule actually applies as a first approximation to any round-nosed airfoil at the ideal angle of attack. Corresponding rules

[5]These values have been verified by comparison with a more extensive table of the coefficients occurring in equation (13) that was kindly communicated by Prof. Imai.

[6]A convergence proof was reported in reference 13, but has never been published.

for lifting and cambered airfoils at other angles of attack can be derived from the Janzen-Rayleigh solution for a parabola with circulation.[7] This can be extracted to order M^2 from existing solutions for inclined ellipses; however, these solutions have not yet been carried as far as the M^4 terms, which are the first to involve γ and are probably important for numerical accuracy.

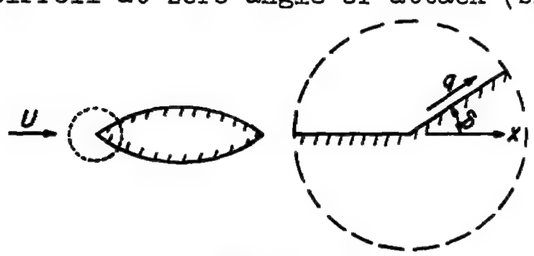
The pressure coefficient must be calculated from the velocities according to the full Bernoulli equation for isentropic flow:

$$C_p = \frac{2}{\gamma M^2}\left\{\left[1 + \frac{\gamma - 1}{2} M^2 \left(1 - \frac{q^2}{U^2}\right)\right]^{\frac{\gamma}{\gamma - 1}} - 1\right\} \tag{17}$$

SHARP-NOSED AIRFOILS

Aside from round noses, sharp edges with finite vertex angle arise most commonly in practice. At sharp edges, thin-airfoil theory predicts logarithmically infinite velocities in place of stagnation points and, for the circulatory component, predicts a leading-edge singularity that exceeds its actual strength. From a practical point of view, it is probably worthwhile to correct these shortcomings only at sharp leading edges because at trailing edges the details will ordinarily be masked by viscous effects. In any case, the correction is significant over a much smaller region for a sharp edge than for a round one.

Consider first incompressible flow past a sharp-nosed symmetrical airfoil at zero angle of attack (sketch (e)). Clearly the flow in the immediate vicinity of the nose is like the flow in an angle.[8] By conformal mapping, or separation of variables in polar coordinates, the surface speed is found to be given by



Sketch (e)

$$\frac{q}{U} = cx^{\frac{\delta}{\pi - \delta}} \tag{18}$$

[7]Because of the nonlinear nature of compressible flow, here the "ideal" and circulatory components can no longer be treated separately, which somewhat complicates the rules. Even in incompressible flow, if the two components are treated together, there results instead of equation (3b) an alternative rule that is more complicated but, of course, entirely equivalent up to terms of second order. In particular, this alternative rule requires that the coefficient k of equations (4) and (5) be known. Hence, in the rule for subsonic flow at other angles of attack than the ideal, it will be necessary to determine k as the limit of the coefficient of $x^{-1/2}$ in the thin-airfoil solution as the airfoil thickness ratio vanishes (corresponding to transition to a true parabola near the nose).

[8]See reference 14 for a rigorous discussion of this point.

where δ is the semivertex angle. In fixing the constant c, the difficulty that in the angle flow the velocity increases indefinitely upstream can be circumvented by requiring that at any point the velocity must approach that of the free stream as the angle δ tends to zero. Thus, it is seen that c is unity except for terms of order δ.

The connection with thin-airfoil theory follows from the fact that for small ν

$$x^{\nu} = e^{\nu \log x} = 1 + \nu \log x + \frac{1}{2}\nu^2 \log^2 x + \ldots \tag{19}$$

though not uniformly near $x = 0$. Hence, the thin-airfoil series for flow in an angle is, to second order,

$$\frac{q}{U} = c\left[1 + \frac{\delta}{\pi} \log x + \frac{\delta^2}{\pi^2}\left(\log x + \frac{1}{2} \log^2 x\right)\right] \tag{20}$$

and it is clear how the spurious logarithmic singularities arise.

Comparing these expressions gives a rule that renders the second-order thin-airfoil solution for any sharp-nosed profile uniformly valid:

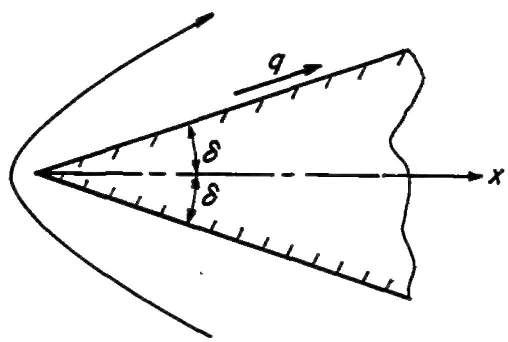
$$\frac{\bar{q}_2}{U} \doteq x^{\frac{\delta}{\pi-\delta}}\left[\frac{"q_2"}{U} - \frac{\delta}{\pi}\frac{"q_1"}{U} \log x - \frac{\delta^2}{\pi^2}\left(\log x - \frac{1}{2} \log^2 x\right)\right] \tag{21a}$$

The corresponding first-order rule is

$$\frac{\bar{q}_1}{U} = x^{\frac{\delta}{\pi-\delta}}\left(\frac{"q_1"}{U} - \frac{\delta}{\pi} \log x\right) \tag{21b}$$

These rules, unlike those for round noses, can be extended to indefinitely high order simply by retaining more terms in equation (20). The reason for this is that the region of nonuniformity is exponentially small, so that the shape of the profile (if it is composed of analytic arcs) enters only through its initial angle.

The second of these rules applies as a first approximation to any sharp-nosed airfoil at its ideal angle of attack. At other angles the flow includes a circulatory component, which behaves near the edge like that around the angle shown in sketch (f). For this component, conformal mapping gives the surface speed as



Sketch (f)

$$\frac{q}{U} = \pm k \sin \alpha_0 \, x^{-\frac{\pi - 2\delta}{2(\pi - \delta)}} = \pm \frac{k \sin \alpha}{\sqrt{x}} \left(1 + \frac{\delta}{2\pi} \log x + \ldots\right) \tag{22}$$

The constant k, which need not be identified, is of order unity, so that logarithmic terms arise in thin-airfoil theory only in the terms of second order in α_0 and δ. The rule for correcting this circulatory component is, to first order,

$$\frac{\bar{q}_1}{U} = x^{\frac{\delta}{2(\pi - \delta)}} \frac{"q_1"}{U} \tag{23a}$$

and to second order

$$\frac{\bar{q}_2}{U} = x^{\frac{\delta}{2(\pi - \delta)}} \left(\frac{"q_2"}{U} - \frac{\delta}{2\pi} \frac{"q_1"}{U} \log x\right) \tag{23b}$$

Note that according to thin-airfoil theory, the circulatory component involves speeds with an inverse square-root singularity, the actual singularity being slightly weaker. Here the rules, in contrast to those for round noses, are different for the two components. Hence, the thin-airfoil solution must be treated by splitting off the terms which are singular at least as $x^{-1/2}$, applying equation (23) to this circulatory component, and then applying equation (21) to the remainder.

It is clear that corners on a profile elsewhere than at the leading or trailing edges will lead to logarithmic singularities in thin-airfoil theory that can, for incompressible flow, be treated by a modification of this procedure.

At subsonic speeds, the flow at the ideal angle of attack could be treated by considering the Janzen-Rayleigh solution for flow in an angle (which has not yet been calculated). However, at other angles the Janzen-Rayleigh method certainly fails, because it would predict infinite velocities that are tolerable only in an incompressible fluid. The present comparison method could be applied only if a truly transonic solution were known for an inclined wedge.

COMBINED EDGES; AN EXAMPLE

The rules derived above for a single edge at the origin can clearly be combined to treat leading and trailing edges together. In order to use the previous results directly it is essential, as Lighthill has pointed out, that the x axis be chosen to pass through both edges. It is also essential to bear in mind the multiplicative nature of the rules.

Thus, consider incompressible flow past an airfoil with round leading and trailing edges of radii ρ_a and ρ_b located (without loss of generality) at $x = -1$ and $x = 1$, respectively. Applying equation (3a) twice, identifying x successively with $(x + 1)$ and $(1 - x)$, and then simplifying to keep no more than second-order terms (recalling that the ρ are of second order) gives

$$\frac{\bar{q}}{U} = \left[\frac{1 - x^2}{1 - x^2 + \frac{1}{2}\rho_a(1 - x) + \frac{1}{2}\rho_b(1 + x)}\right]^{1/2}\left(\frac{"q_2"}{U} + \frac{1}{4}\frac{\rho_a}{1 + x} + \frac{1}{4}\frac{\rho_b}{1 - x}\right) \tag{24}$$

(Note that the simplification has included dropping a term $\rho_a\rho_b/4$ in the denominator of the radical, since its contribution can be shown to be of higher order.) This rule converts the second-order thin-airfoil solution into a uniformly valid second approximation (in the sense discussed previously) for airfoils whose initial and final camber is zero, but gives only a first approximation near a cambered edge.

In the same way the previous rules can be combined for higher approximations and for any combination of round and sharp edges.

As an application of the rule for combined round edges, consider incompressible flow past an ellipse of thickness ratio ϵ at zero angle of attack. The formal second-order solution for surface speed is

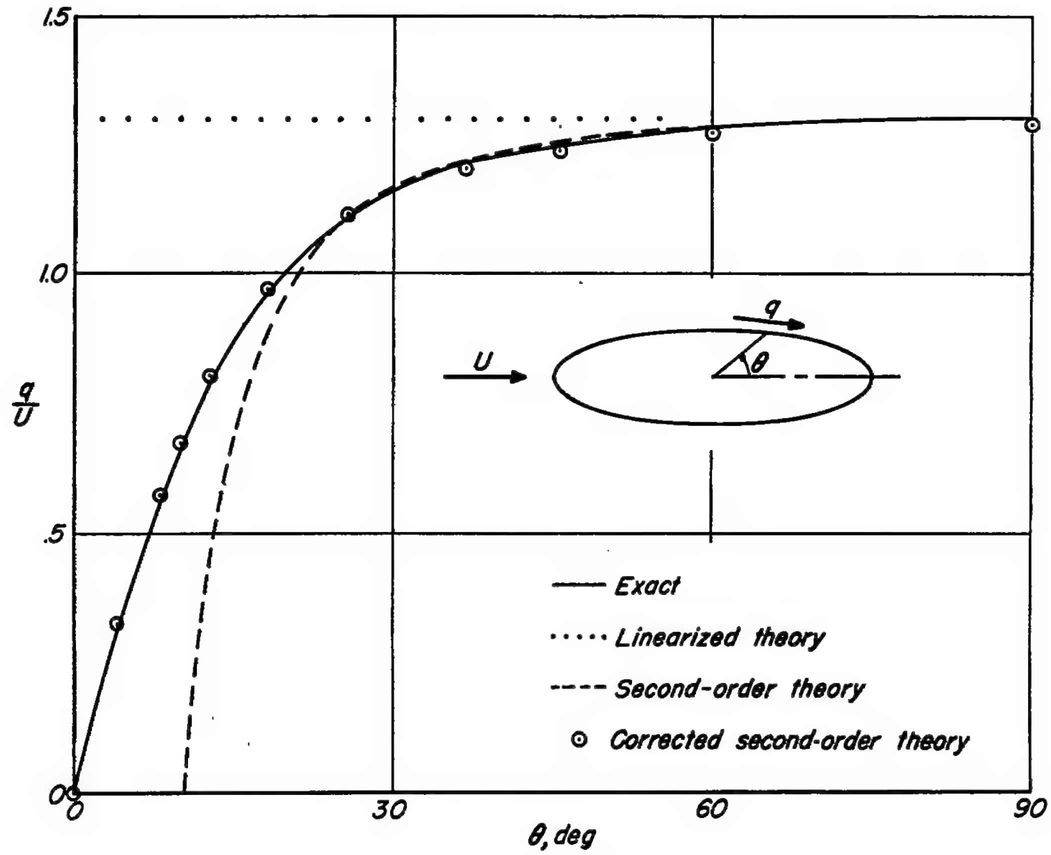
$$\frac{"q_2"}{U} = 1 + \epsilon - \frac{1}{2}\epsilon^2\frac{x^2}{1 - x^2} \tag{25}$$

Applying the rule of equation (24), with $\rho_a = \rho_b = \epsilon^2$, gives

$$\frac{\bar{q}_2}{U} = \left(\frac{1 - x^2}{1 - x^2 + \epsilon^2}\right)^{1/2}\left(1 + \epsilon + \frac{1}{2}\epsilon^2\right) \tag{26}$$

and it is satisfying to see that aside from the radical, the expression

has been simplified. This result is compared in sketch (g) with the



Sketch (g)

exact solution and with the formal first- and second-order solutions for an ellipse of thickness ratio $\epsilon = 3/10$.

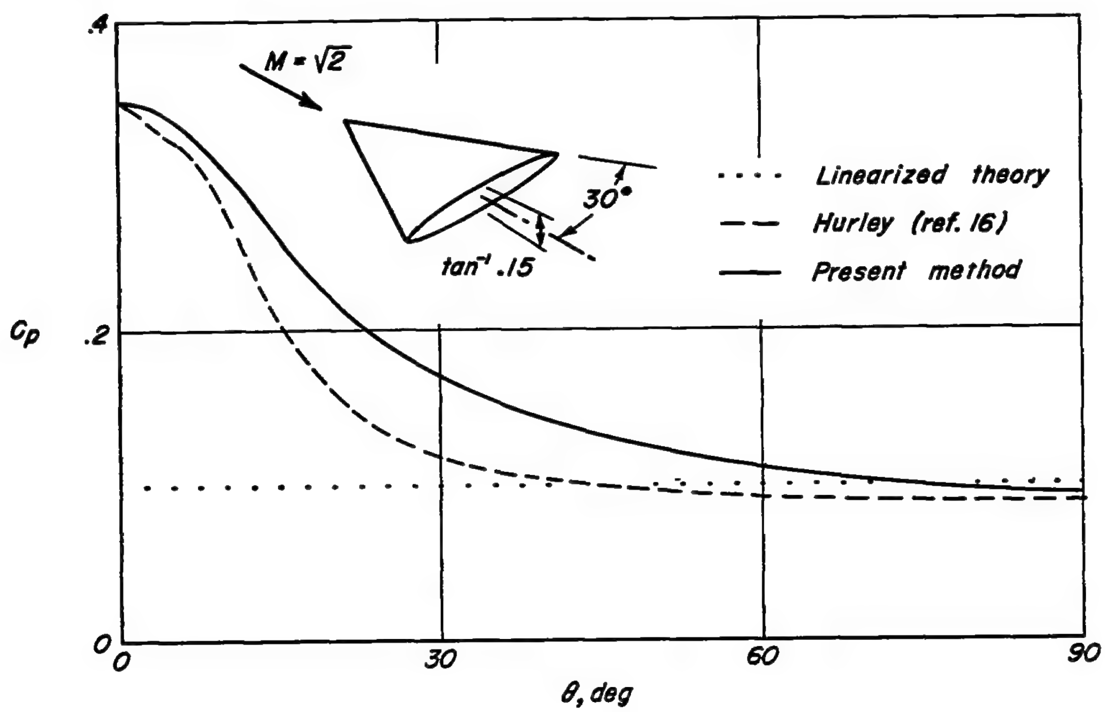
APPLICATION TO FINITE WINGS

The edges of a finite wing are described as subsonic or supersonic according as the component of free-stream velocity normal to them is subsonic or supersonic. At subsonic edges thin-wing theory breaks down just as it does for two-dimensional airfoils in subsonic flow.[9] These defects can be corrected by applying the preceding rules to the components of velocity in the plane normal to the edge; for, as noted by R. T. Jones (ref. 2), for sufficiently thin wings, the flow field in the immediate vicinity of the edge is cylindrical (if the plan form and edge radius are continuous) according to the simple sweepback concept. Since thin-wing

[9] Thin-wing theory fails also at round supersonic edges. This defect could probably be corrected by comparing with the solution for supersonic flow past a parabola, which has not yet been calculated.

theory is a series expansion in powers of thickness, its coefficients can be evaluated at vanishing thickness. Hence, the sweepback principle applies here to any order.

As an example, consider supersonic flow past a flat elliptic cone lying inside the free-stream Mach cone. The linearized thin-wing solution, first given by Squire (ref. 15), predicts constant pressure over the entire surface. Hurley has attempted to improve this result near the leading edges by adding, along the foci of the ellipse, conical line sources whose strength is determined by imposing the exact tangency condition at the leading edge (ref. 16). He has verified that the resulting shape is approximately the desired one, though somewhat sharper near the leading edge. For the thickest wing that he considered, Hurley's pressure distribution is compared in sketch (h) with the first-order result of the present method. The present solution is presumably more accurate since the shape is not distorted and compressibility effects are accounted for more exactly.



Sketch (h)

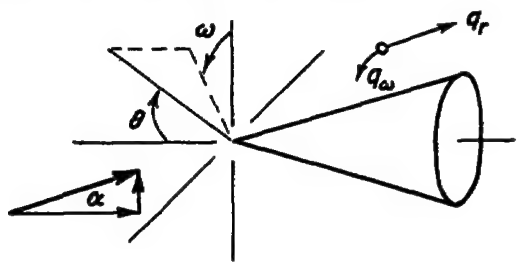
BODIES OF REVOLUTION

For bodies of revolution, the counterpart of thin-wing theory is the slender-body theory introduced by Munk for the crossflow due to angle of attack, and later extended to cover also the flow at zero angle. For subsonic speeds this theory, too, fails near stagnation points, where it

predicts infinite rather than zero speeds. Furthermore, for the crossflow it predicts a nonexistent singularity at round ends and fails to predict an actual singularity at sharp ends. It can be corrected by simple rules quite analogous to those already given for wings. Only incompressible flow will be considered here, the extension to subsonic speeds requiring the solution for simple nose shapes according to the Janzen-Rayleigh method or otherwise.

Any round-nosed body of revolution can be approximated near its nose by a paraboloid of revolution. Munk's rule gives the remarkable result that the surface speeds for axial flow are identical on parabolas and paraboloids of revolution, and that the additional velocities due to angle of attack are proportional. Hence, Lighthill's rule for round-nosed airfoils, equation (3b), applies also to bodies of revolution. It yields, however, only a first approximation because the disturbances, being smaller than for airfoils, are of the order of the nose radius (i.e., of order ϵ^2) rather than of order ϵ. A second-order rule can be obtained by fitting the nose more accurately with an ellipsoid of revolution. Again, Munk's rule shows that the speed distributions are proportional on ellipses and ellipsoids of revolution of the same thickness ratio. Hence, the fourth-order rule for airfoils, equation (11), is the second-order rule for round-nosed bodies of revolution.

For sharp-nosed bodies, the nature of the flow near the tip can be found by introducing spherical polar coordinates (sketch (i)). Separating variables leads to a solution in terms of Legendre functions for the axial flow and associated Legendre functions for the crossflow, the functions being of nonintegral order. Requiring, as in plane flow, that the solution reduce to a uniform stream as the cone angle δ vanishes shows that on the surface in the immediate vicinity of the tip the radial velocity component has the form



Sketch (i)

$$\frac{q_r}{U} = c \cos \alpha \, x^{m-1} + k \sin \alpha \cos \omega \, x^{n-1} \tag{27a}$$

and the azimuthal component, the form

$$\frac{q_\omega}{U} = -k \sin \alpha \sin \omega \, x^{n-1} \tag{27b}$$

Here, again, c is a constant that is unity to first order and k is a constant of order unity. The tangency condition requires that m and n be the first roots of

$$\left.\begin{aligned} P_m{}'[\cos(\pi - \delta)] &= 0 \\ P_n{}^{1\prime}[\cos(\pi - \delta)] &= 0 \end{aligned}\right\} \tag{28a}$$

where the derivatives of the Legendre functions are considered as functions of m and n. Methods for calculating these roots numerically have been discussed by Pal (ref. 17) and by Siegel et al. (ref. 18). Using a formula due to Schelkunoff (ref. 19), it can be shown that to a first approximation for small cone angle δ

$$\left.\begin{aligned} m &= 1 + \frac{1}{2}\delta^2 + O(\delta^4 \log \delta) \\ n &= 1 - \frac{1}{2}\delta^2 + O(\delta^4 \log \delta) \end{aligned}\right\} \tag{28b}$$

Since $n - 1$ is slightly negative, the velocities associated with the crossflow are only weakly infinite at the tip, in contrast to the nearly inverse square-root singularity associated with angle of attack for a sharp-nosed airfoil.

Expanding equation (27a) according to equation (19) leads again to rules for correcting slender-body theory. As in plane flow, the axial flow and crossflow must be treated separately. The rule for correcting the axial component is found to be, to first order,

$$\frac{\tilde{q}_1}{U} = x^{m-1}\left(\frac{"q_1"}{U} - \frac{1}{2}\delta^2 \log x\right) \tag{29a}$$

This could be extended to second order by determining the next term in the series for m. The rule for correcting either the radial or azimuthal component of velocity associated with the crossflow is, to second order,

$$\frac{\tilde{q}_2}{U} = x^{n-1}\left(\frac{"q_2"}{U} + \frac{1}{2}\delta^2 \log x\right) \tag{29b}$$

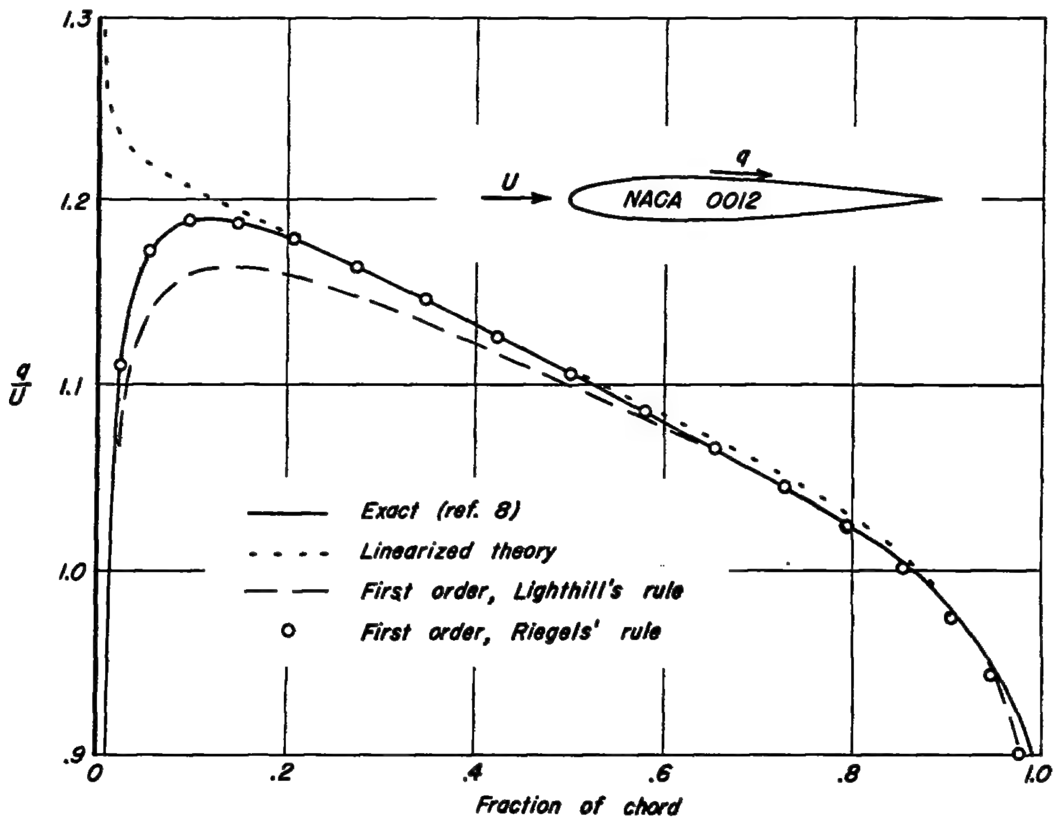
The corresponding first-order rule is the same without the logarithmic term.

ALTERNATIVE FORMS OF RULES FOR ROUND NOSES

The radical $\left(\frac{x}{x + \rho/2}\right)^{1/2}$ appearing in the rules for round noses in incompressible flow is, for a parabola or paraboloid, the cosine of the surface angle η measured from the x axis. Therefore, the first-order rule for airfoils, equation (8), can be written alternatively as

$$\frac{\bar{q}_1}{U} = \cos\eta\left(\frac{"q_1"}{U}\right) = \frac{1}{\sqrt{1 + \tan^2\eta}}\,\frac{"q_1"}{U} \tag{30}$$

and this is Riegels' rule (ref. 9). Aside from its simplicity, it has an unexpected advantage over Lighthill's form. The two rules differ only by higher-order terms for shapes other than a parabola, so that there is no a priori reason to prefer either. However, it happens that Riegels' form gives the exact result for an ellipse as well as for a parabola,[10] whereas Lighthill's does not. Since airfoil profiles ordinarily resemble ellipses more than parabolas, Riegels' form might be expected to be more accurate, and this appears to be the case. For example, sketch (j) shows that Riegels' rule gives a much more accurate first-order solution than Lighthill's for an NACA 0012 airfoil at zero angle of attack.



Sketch (j)

[10]This is clear from Munk's rule (together with the fact that linearized theory predicts a constant speed on an ellipse that is equal to the true maximum value).

Despite its simplicity and remarkable accuracy, however, Riegels' rule must be regarded as less basic than Lighthill's from a theoretical point of view. Only Lighthill's rule can, in principle, be extended to indefinitely high order, to subsonic speeds, or to other nose shapes. A further defect of Riegels' rule is that the nose exerts a spurious influence even at remote points if the local airfoil slope is appreciable. To this extent it fails to render the solution uniformly valid.

Nevertheless, the practical advantages of Riegels' rule are so great as to suggest artificially modifying the higher-order rules accordingly. Thus, Lighthill's second-order rule for uncambered airfoils, equation (3b), can be rewritten by analogy with Riegels' rule as

$$\frac{\bar{q}_2}{U} = \cos\eta\left(\frac{"q_2"}{U} + \frac{1}{2}\tan^2\eta\right) \tag{31}$$

Both forms are so accurate that graphical comparison is scarcely possible. Instead, they are compared in the following table with the exact values for a 14-percent-thick symmetrical Joukowski airfoil at zero angle of attack:

Fraction of chord	Second order, Lighthill	Second order, alternative	Exact (ref. 20)
0.005	0.7530	0.7363	0.7379
.0075	.8590	.8440	.8449
.0125	.9852	.9748	.9738
.025	1.1249	1.1226	1.1190
.05	1.2137	1.2185	1.2132
.10	1.2489	1.2568	1.2524

The superiority of the alternative form is now confined to the immediate vicinity of the nose (which has a radius of 0.023 chord). Similarly, for cambered airfoils, the second-order rule can be recast in the form

$$\frac{\bar{q}_2}{U} = \cos(\eta - \kappa)\left[\frac{"q_2"}{U} + \frac{1}{2}\tan^2(\eta - \kappa)\right] \tag{32}$$

where κ is the initial angle of the camber line; and comparison with its counterpart in equation (10) illustrates the greater simplicity of the alternative forms, which is perhaps their chief virtue in the higher-order approximations.

Again, both these rules are exact for ellipses, whereas their predecessors are not. Since this is so, the fourth-order rule (which results from considering ellipses) is obtained simply by retaining another term

in the formal series expansion of the cosine. For initially uncambered airfoils this leads to

$$\frac{\bar{q}_4}{U} = \cos \eta \left(\frac{"q_4"}{U} + \frac{1}{2} \frac{"q_2"}{U} \tan^2 \eta - \frac{1}{8} \tan^4 \eta \right) \tag{33}$$

and it can be verified that up to terms of fourth order this agrees with the less compact form of equation (11). The corresponding rule for cambered airfoils is

$$\frac{\bar{q}_4}{U} = \cos(\eta - \kappa)\left[\frac{"q_4"}{U} + \frac{1}{2} \frac{"q_2"}{U} \tan^2(\eta - k) - \frac{1}{8} \tan^4(\eta - \kappa)\right] \tag{34}$$

For round-nosed bodies of revolution, equation (31) is the first-order rule and equation (33) the second-order rule. These are exact for ellipsoids of revolution as well as paraboloids. It can therefore be anticipated that they will be generally more accurate than the previous forms.

The rule for subsonic speeds can also be artificially manipulated in an attempt to preserve the advantages of Riegels' rule. Thus, equation (16a) is replaced by

$$\frac{\bar{q}_2}{U} = \tilde{Q}(\eta) \left[\frac{"q_2"}{U} + \frac{1}{2}\left(1 + \frac{M^2}{\beta^2} + \frac{\gamma + 1}{4} \frac{M}{\beta}\right) \tan^2 \eta\right] \tag{35}$$

where $\tilde{Q}(\eta)$ is the subsonic speed ratio on a parabola expressed in terms of the surface angle η. It is related to the function $Q(x/\rho)$ of equations (15) by

$$\tilde{Q}(\eta) = Q\left(\frac{1}{2} \cot^2 \eta\right) \tag{36}$$

because for a parabola $\rho/x = 2 \tan^2 \eta$. This rule differs from its predecessor only by higher-order terms and is designed to reduce to equation (31) at zero Mach number.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., Aug. 16, 1954

APPENDIX

SYMBOLS

C_p	pressure coefficient (pressure increment divided by free-stream dynamic pressure)
C_1	initial slope of camber line divided by ϵ
c	coefficient that is unity to first order
$F_0, F_1, \ldots$	coefficients in expansion of airfoil thickness
f_0, f_1, f_2, f_3	coefficients of Janzen-Rayleigh expansion for Q
k	coefficient of order unity
M	free-stream Mach number
m	first root of $P_m'[\cos(\pi - \delta)] = 0$, where $P_m(\cos\theta)$ is the Legendre function
n	first root of $P_n^{1\prime}[\cos(\pi - \delta)] = 0$, where $P_m^1(\cos\theta)$ is the associated Legendre function
Q	speed ratio on parabola in subsonic axial flow as function of x/ρ
$\tilde{Q}$	speed ratio on parabola in subsonic axial flow as function of surface angle η
q	speed at body surface
$\bar{q}$	uniformly valid approximation to q
"q"	formal thin-wing or slender-body solution for q
$q_1, q_2, \ldots$	first-order, second-order, etc., approximation to q
U	free-stream speed
u	component of surface velocity in x direction
x	coordinate along airfoil chord or body axis, usually measured from edge
y	coordinate of airfoil surface perpendicular to chord
α	angle of attack
α_0	angle of attack measured from ideal angle

β	$(1 - M^2)^{1/2}$
γ	adiabatic exponent of gas
δ	semivertex angle of pointed airfoil or body
ϵ	thickness parameter of airfoil
η	surface angle measured from x axis
θ	polar angle
κ	initial angle of camber line
λ	(see eq. (11))
ρ	edge radius of round-ended airfoil or body
ω	azimuthal angle

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